Flux-balance laws from the Hamiltonian formulation of self-force

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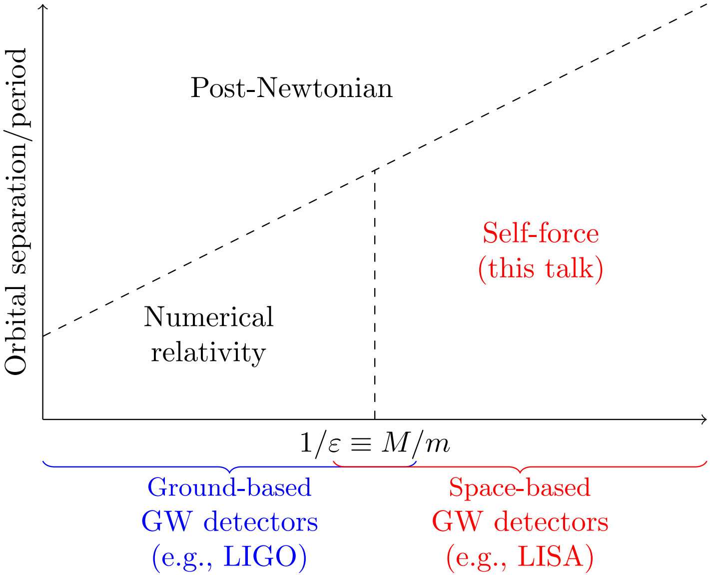
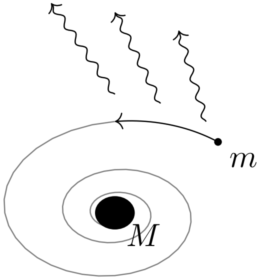
Motivation: the gravitational two-body problem



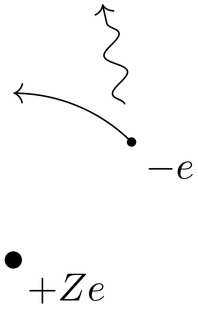


Example: self-force in classical electromagnetism

- Self-force arises in "classical atom":
  local EM field of orbiting charge $\implies$ self-force
- Instead of local field, use "global energy balance":

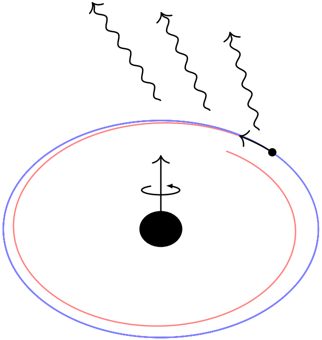
$$\overbrace{\left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle}^{\substack{\text{rate of change of}\\ \text{conserved quantity}}} = \underbrace{-\langle P \rangle}_{\substack{\text{flux of}\\ \text{conserved current}}} < 0 \quad \implies \quad \text{orbit decays}$$

- Common features w/ gravitational self-force:
  - Local fields hard to work with (must be regularized), but flux *only* depends on fields far away
  - Average cancels out energy stored locally in field



Motion around spinning black holes

- Unforced, geodesic motion in Kerr:
  complete set of conserved quantities P_α
  - Linear in p_a: from Killing vectors/isometries:
    - $E \equiv -(\partial_t)^a p_a$
    - $L_z \equiv (\partial_\phi)^a p_a$
  - Quadratic in p_a: from "Killing tensors" obeying $\nabla_{(a} K_{bc)} = 0$:
    - $m^2 \equiv -g_{ab} p^a p^b$
    - "Carter constant" $K \equiv K_{ab} p^a p^b$ (reduces to L^2 in Schwarzschild)
- Non-geodesic motion: "conserved quantities" $P_\alpha(\tau)$
  (flux-balance gives you $\langle \mathrm{d}P_\alpha / \mathrm{d}\tau \rangle$)



Previous work on flux-balance laws

Properties of Green('s) functions[a]

- [Gal'tsov, 1982]: E & L_z (scalar, electromagnetism, & gravity)
- [Mino, 2003]: Carter constant K (gravity)
- [Isoyama et al, 2018]: any action variable (gravity)

[a] No physically motivated explanation

Conserved currents

- [Quinn & Wald, 1999]: E & L_z (scalar, electromagnetism, & gravity, *only scattering*)
- [**Grant** & Moxon, 2022]: any action variable (scalar field, *only locally*)
- This talk: any action variable *or* conserved quantity (gravity)

Outline

I. Conserved currents

II. Hamiltonian formulation of self-force

III. Flux-balance laws

Local variational principles

- Consider theory for field Φ_A, w/ equations of motion $E^A = 0$
- Usual way of deriving these equations:

$$S = \int_V L \, \mathrm{d}V$$

$$\Downarrow$$

$$\underbrace{\delta S = 0}_{\substack{\text{for all } \delta\Phi_A \\ \text{subject to B.C.}}} \iff E^A = 0$$

- This throws out integral over ∂V, so consider a *local* expression:

$$\delta(\sqrt{-g}L) = \sqrt{-g}\left(E^A \delta\Phi_A + \nabla_a \theta^a \{\delta \boldsymbol{\Phi}\}\right)$$

for local, linear functional θ^a

Symplectic currents

- In terms of θ^a, define *symplectic current*

$$\omega^a\{\delta_1\boldsymbol{\Phi},\delta_2\boldsymbol{\Phi}\} \equiv \frac{1}{\sqrt{-g}}\delta_1(\sqrt{-g}\theta^a\{\delta_2\boldsymbol{\Phi}\}) - \delta_1 \longleftrightarrow \delta_2$$

- Conserved on *linearized* EOM for perturbations $\delta_i\Phi$:

$$\nabla_a\omega^a\{\delta_1\boldsymbol{\Phi},\delta_2\boldsymbol{\Phi}\} = \frac{1}{\sqrt{-g}}\delta_1\Phi_A \underbrace{\delta_2(\sqrt{-g}E^A)}_{\sqrt{-g}\underset{(1)}{E}{}^A\{\delta_2\boldsymbol{\Phi}\}} - \delta_1 \longleftrightarrow \delta_2$$

- In fact, shows $\underset{(1)}{E}{}^A$ is self-adjoint!

Example: scalar field

- (Real) Klein-Gordon Lagrangian:

$$\left.\begin{aligned} &\Phi = \phi, \\ &L = -\frac{1}{2} g^{ab} (\nabla_a \phi)(\nabla_b \phi) \end{aligned}\right\} \implies \begin{cases} E = \Box \phi, \\ \theta^a\{\delta\phi\} = -\delta\phi\, g^{ab} \nabla_b \phi \end{cases}$$

- Symplectic current just analogue of Klein-Gordon current:

$$\omega^a\{\delta_1\phi, \delta_2\phi\} = -g^{ab}(\delta_2\phi \nabla_b \delta_1\phi - \delta_1 \longleftrightarrow \delta_2)$$

- Theory is *linear* $\implies$ $\delta_1\phi \equiv \phi_1$ and $\delta_2\phi \equiv \phi_2$ exact solutions

Example: gravity

- Einstein-Hilbert Lagrangian:

$$\left.\begin{aligned} \Phi_{ab} &= g_{ab}, \\ L &= -R \end{aligned}\right\} \implies \begin{cases} E^{ab} = G^{ab}, \\ \theta^a\{\delta \boldsymbol{g}\} = -2\,\underbrace{\delta C^{[a}{}_{bc}}_{\text{varied Christoffels}}\, g^{b]c} \end{cases}$$

- Symplectic current:

$$\omega^a\{\delta_1 \boldsymbol{g}, \delta_2 \boldsymbol{g}\} = -2\left(\delta_2 C^{[a}{}_{bc} \delta_1 g^{b]c} + \frac{1}{2} g^{de} \delta_1 g_{de} \delta_2 C^{[a}{}_{bc} g^{b]c}\right) - \delta_1 \longleftrightarrow \delta_2$$

- Gauge-invariant up to boundary term:

$$\omega^a\{\delta \boldsymbol{g}, \pounds_\xi \boldsymbol{g}\} = \nabla_b Q^{[ab]}$$

Symmetry operators

- Symmetry operator $\mathcal{D}^A{}_B$:

$$\underset{(1)}{E}^A\{\boldsymbol{\mathcal{D}} \cdot \delta\boldsymbol{\Phi}\} = \widetilde{\mathcal{D}}^A{}_B \underset{(1)}{E}^B\{\delta\boldsymbol{\Phi}\} \implies \begin{array}{c}\mathcal{D}^A{}_B \text{ maps b/w solutions} \\ \text{of linearized equations}\end{array}$$

- Quadratic conserved current $\omega^a\{\delta\boldsymbol{\Phi}, \boldsymbol{\mathcal{D}} \cdot \delta\boldsymbol{\Phi}\}$
- Examples:
  - For isometry/Killing vector ξ^a, $\pounds_\xi$ (for *any* field)
  - For Killing tensor K_{ab} and Klein-Gordon field ϕ,

$$\mathcal{D}_K : \phi \mapsto \nabla_a(K^{ab}\nabla_b\phi) \qquad \text{[Carter, 1977]}$$

(others for other fields in Kerr, see, e.g., [**Grant** & Flanagan, 2019 & 2020])

- This talk: symmetry operator comes from Hamiltonian formulation

Outline

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“Axioms” of first-order, gravitational self-force

1. Exact worldline $\gamma(\varepsilon)$ geodesic in $\breve{g}_{ab}(\varepsilon) = g_{ab} + \varepsilon h^R_{ab} + O(\varepsilon^2)$:

$$\dot{\gamma}^b(\varepsilon)\breve{\nabla}_b(\varepsilon)\dot{\gamma}^a(\varepsilon) = O(\varepsilon^3), \qquad \dot{\gamma}^a(\varepsilon)\dot{\gamma}^b(\varepsilon)\breve{g}_{ab}(\varepsilon) = -1,$$

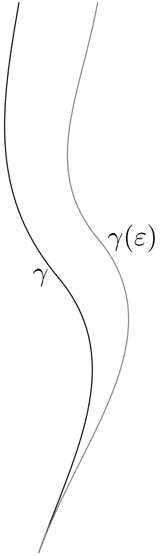
2. Full retarded field: $h^1_{ab} = h^R_{ab} + h^S_{ab}$ obeys

$$\underbrace{\underset{(1)}{E}{}^{ab}\{\boldsymbol{h}^1\} = 0}_{\text{off } \gamma} \qquad \text{and} \qquad \underbrace{\begin{matrix} \underset{(1)}{E}{}^{ab}\{\boldsymbol{h}^R\} = 0 \\ \underset{(1)}{E}{}^{ab}\{\boldsymbol{h}^S\} = 8\pi T^{ab}_1 \end{matrix}}_{\text{near } \gamma}$$

3. Stress-energy tensor:

$$\sqrt{-g}T^{ab}_1(x) = m\int \mathrm{d}\tau' \dot{\gamma}^{a'}\dot{\gamma}^{b'}\delta^{ab}{}_{a'b'}[x,\gamma(\tau')], \text{ where}$$

$$\int_V f_{ab}\delta^{ab}{}_{a'b'}(x,x')\mathrm{d}V = \begin{cases} f_{a'b'} & x' \in V \\ 0 & x' \notin V \end{cases}$$



Self force as a Hamiltonian system

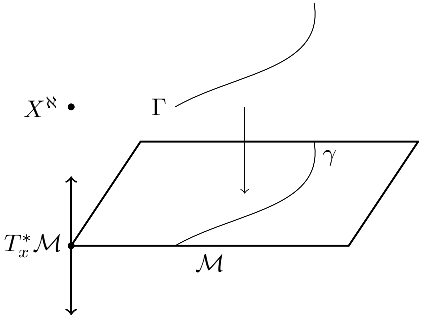
- Points on phase space: $X^{\aleph} = \binom{x^\alpha}{p_\alpha}$
- $\gamma(\varepsilon)$ geodesic in $\breve{g}_{ab}(\varepsilon) \implies$

$$H(X;\varepsilon) = -\sqrt{-\breve{g}^{\alpha\beta}(x;\varepsilon)p_\alpha p_\beta} + O(\varepsilon^3)$$

- "Velocity" on phase space: Hamilton's equations

$$\dot{\Gamma}^A(\varepsilon) = \underbrace{(\Omega^{-1})^{AB}}_{\text{Poisson bracket}} \nabla_B H(\varepsilon)$$

(valid even if there is no symplectic form Ω_{AB}!)



Pseudo-Hamiltonians

- Hamiltonian systems are supposed to be conservative, but self-force isn't—what's going on?

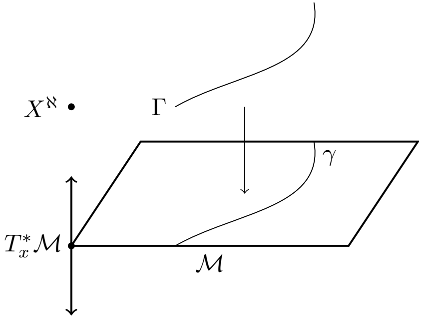
- Note: $\breve{g}_{ab}$ *itself* depends on a worldline, so really

$$\underbrace{H(X, \bar{X}; \varepsilon)}_{\text{“pseudo-Hamiltonian”}} = -\sqrt{-\breve{g}^{\alpha\beta}[x; \Upsilon(\bar{X})] p_\alpha p_\beta}$$

where $\Upsilon : X \mapsto \Gamma$ such that

$$\dot{\Gamma}^A(\varepsilon) = (\Omega^{-1})^{AB} [\nabla_B H(X, \bar{X}; \varepsilon)]_{\bar{X} \to X}$$

- Becomes Hamiltonian for $\varepsilon = 0$ (geodesic motion)



Perturbation theory

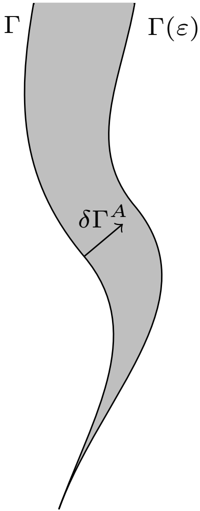
- Quantity of interest: $\delta\Gamma^A$, the tangent to $\Gamma(\tau;\varepsilon)$ *at constant* τ, $\varepsilon = 0$

- Evolution:

$$\pounds_{\dot{\Gamma}}\delta\Gamma^A = (\Omega^{-1})^{AB}[\nabla_B \delta H(X,\bar{X})]_{\bar{X}\to X}$$

- Given a $\Upsilon^{A'}{}_A$ obeying $\pounds_{\dot{\Gamma}'}\Upsilon^{A'}{}_A = 0$, can define an average rate of change of $\delta\Gamma^A$:

$$\left\langle \delta\dot{\Gamma}^A \right\rangle \equiv \lim_{\Delta\tau\to\infty} \frac{\Upsilon^A{}_{A''}\delta\Gamma^{A''} - \Upsilon^A{}_{A'}\delta\Gamma^{A'}}{\Delta\tau}$$
$$= (\Omega^{-1})^{AB}\left\langle \Upsilon^{B'}{}_B \left[\nabla_{B'}\delta H(X',\bar{X}')\right]_{\bar{X}'\to X'} \right\rangle_{\tau'}$$



Hamilton propagator

- For fixed τ, τ', consider

$$\Upsilon(\tau,\tau') : \underbrace{\Gamma(\tau)}_{X} \mapsto \underbrace{\Gamma(\tau')}_{X'}$$

- *Pushforward* ${\Upsilon^{A'}}_A$ relates vectors at X & X', defined by

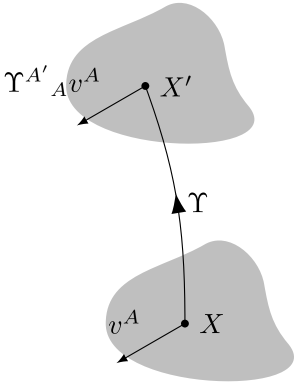
$$\nabla_A f[\Upsilon(X)] \equiv {\Upsilon^{A'}}_A \nabla_{A'} f(X')\big|_{X'=\Upsilon(X)}$$

- Also obeys

$$\pounds_{\dot{\Gamma}'} {\Upsilon^{A'}}_A = 0$$

- In coordinates:

$$\Upsilon^{\aleph}{}_{\beth}(\tau',\tau) = \frac{\partial X^{\aleph}(\tau')}{\partial X^{\beth}(\tau)}$$

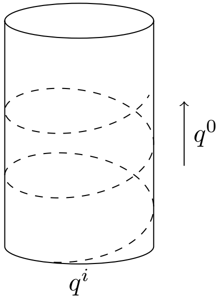


In (action angle) coordinates

- Geodesic motion in Kerr is *integrable*:
  can find independent conserved quantities P_α
- Can write coordinates $X^{\aleph} = \binom{q^\alpha}{J_\alpha}$ such that
  - J_α are conserved
  - q^0 is non-compact, $q^1, \ldots, q^3$ periodic in 2π
  - q^α, J_α canonical: $(\Omega^{-1})^{AB} = 2(\partial_{q^\alpha})^{[A}(\partial_{J_\alpha})^{B]}$
- Hamilton propagator: in terms of frequencies ν^α,

$$\Upsilon^{\aleph}{}_{\beth}(\tau', \tau) = \begin{pmatrix} \delta^\alpha{}_\beta & (\tau' - \tau)\frac{\partial \nu^\alpha}{\partial J_\beta} \\ 0 & \delta^\beta{}_\alpha \end{pmatrix}$$

[similar result holds if using $X^{\aleph} = \binom{q^\alpha}{P_\alpha}$,
but now no longer canonical!]



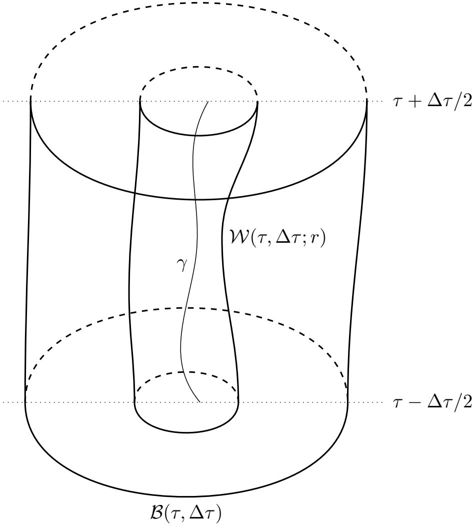
Surface of constant J_α

Outline

Integration regions

- $\mathcal{B}(\tau, \Delta\tau)$: approaches horizon ($\mathscr{H}$) and null infinity ($\mathscr{I}$)
- $\mathcal{W}(\tau, \Delta\tau; r)$: surface of proper distance r (near γ so that this & $h_{ab}^{R,S}$ well-defined)
- Note: computing averages $\lim_{\Delta\tau\to\infty} \frac{1}{\Delta\tau} \int_{\dots}$

  $\implies$ endcap contributions vanish!
  ("conserved quantity stored in field")



A new symmetry operator

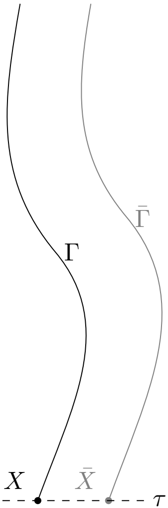
- Fields which we consider are functions of Γ:

$$\Gamma(\tau) \implies T_1^{ab} \implies h^1_{ab}, h^S_{ab}, h^R_{ab}$$

- For fixed τ, Γ is a function of its initial data X at τ (through map Υ)

- New symmetry operator: ∇_A (varies X at fixed τ)

- Note: *only* works on the fields in this problem!



A local flux-balance law

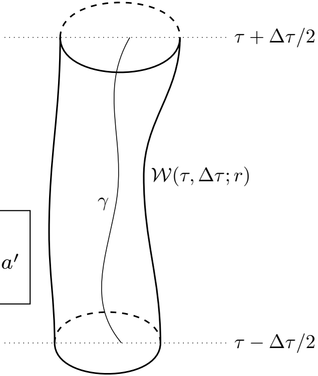
- Using linearized EOM:

$$\nabla_{a'}\omega^{a'}\{\boldsymbol{h}^R, \nabla_A \boldsymbol{h}^S\} = 8\pi h^R_{a'b'} \nabla_A T_1^{a'b'}$$

- Lengthy calculation + Stokes' theorem $\implies$

$$\boxed{\left\langle \delta\dot{\Gamma}^A \right\rangle = -\lim_{\Delta\tau\to\infty} \frac{(\Omega^{-1})^{AB}}{16\pi\Delta\tau} \int_{\mathcal{W}(\tau,\Delta\tau;r)} \omega^{a'}\{\boldsymbol{h}^R, \nabla_B \boldsymbol{h}^S\} \mathrm{d}\Sigma_{a'}}$$

- $h^{R,S}_{ab}$ defined locally $\implies$ works only on $\mathcal{W}(\tau, \Delta\tau; r)$!



A local-to-global approach

Start with

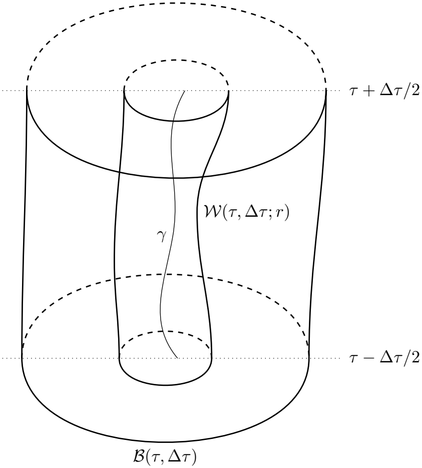
$$\mathcal{F}_A[\boldsymbol{h}^1] \equiv \lim_{\Delta\tau\to\infty} \frac{1}{\Delta\tau} \int_{\mathcal{B}(\tau,\Delta\tau)} \omega^{a'}\{\boldsymbol{h}^1, \nabla_A \boldsymbol{h}^1\} \mathrm{d}\Sigma_{a'}$$

1. In region b/w $\mathcal{B}(\tau, \Delta\tau)$ and $\mathcal{W}(\tau, \Delta\tau; r)$,

$$\nabla_{a'}\omega^{a'}\{\boldsymbol{h}^1, \nabla_A \boldsymbol{h}^1\} = 0$$

$$\implies \int_{\mathcal{B}(\tau,\Delta\tau)} = \int_{\mathcal{W}(\tau,\Delta\tau;r)}$$

2. Decompose $\boldsymbol{h}^1$ at $\mathcal{W}(\tau, \Delta\tau; r)$; by bilinearity, gives
   - $\boldsymbol{h}^R$, $\nabla_A \boldsymbol{h}^R$: exactly conserved, give nothing
   - $\boldsymbol{h}^R$, $\nabla_A \boldsymbol{h}^S$: see previous slide, gives *local* flux-balance
   - $\boldsymbol{h}^S$, $\nabla_A \boldsymbol{h}^R$: not *exactly* the same as previous slide...
   - $\boldsymbol{h}^S$, $\nabla_A \boldsymbol{h}^S$: naïvely diverges



Asymmetry of conserved current

3. $\omega^{a'}\{\boldsymbol{h}, \nabla_A \tilde{\boldsymbol{h}}\}$ not symmetric under $\boldsymbol{h} \longleftrightarrow \tilde{\boldsymbol{h}}$, but:

- From $\nabla_{a'}\boldsymbol{\omega}^{a'}\{\boldsymbol{h}^S, \nabla_A \boldsymbol{h}^R\} = -8\pi T_1^{a'b'}\nabla_A h^R_{a'b'}$, can show

$$\begin{aligned}\lim_{\Delta\tau\to\infty} \frac{(\Omega^{-1})^{AB}}{8\pi\Delta\tau}\int_{\mathcal{W}(\tau,\Delta\tau;r)} \omega^{a'}\{\boldsymbol{h}^S, \nabla_B \boldsymbol{h}^R\}\mathrm{d}\Sigma_{a'} \\ = 2(\Omega^{-1})^{AB}\left\langle \Upsilon^{B'}{}_B[\underbrace{\nabla_{\bar{B}'}}_{\text{previously } \nabla_{B'}}\delta H(X', \bar{X}')]_{\bar{X}'\to X'}\right\rangle_{\tau'}\end{aligned} \qquad (*)$$

- Synge's rule:

$$\nabla_A[f(X,X')]_{X'\to X} = [\nabla_A f(X,X')]_{X'\to X} + [\nabla_{A'} f(X,X')]_{X'\to X}$$

$$\Downarrow$$

$$(*) = \underbrace{-2\left\langle\delta\dot{\Gamma}^A\right\rangle}_{\text{what we want}} + \underbrace{2(\Omega^{-1})^{AB}\left\langle\Upsilon^{B'}{}_B\nabla_{B'}[\delta H(X',\bar{X}')]_{\bar{X}'\to X'}\right\rangle_{\tau'}}_{\text{vanishes off resonance [Isoyama et al., 2018]?}}$$

Resolving the "divergent" piece

4. Can show divergent piece vanishes by parity argument
   - Form of h^S_{ab} near worldline:
   $$\boldsymbol{h}^S \sim m/r$$
   - Integrand contains odd # of n_a's ($\equiv \nabla_a r$), and $\int \mathrm{d}\Omega \; n^{a_1} \cdots n^{a_{2k+1}} = 0$

Final, simple flux-balance law:

$$\boxed{\left\langle \delta\dot{\Gamma}^A \right\rangle = -\frac{1}{32\pi}(\Omega^{-1})^{AB}\mathcal{F}_B[\boldsymbol{h}^1]}$$

Explicit result in coordinates

- For J_α, result simplifies as q^α, J_α canonical:

$$\left\langle \delta \dot{J}_\alpha \right\rangle = \frac{1}{32\pi} (\partial_{q^\alpha})^A \mathcal{F}_A[\boldsymbol{h}^1]$$

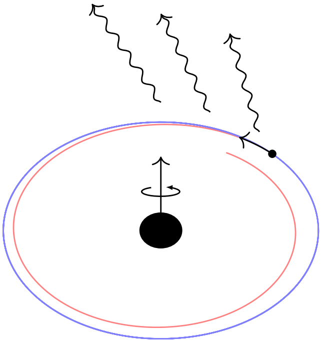
- Compute h^1_{ab} asymptotically and differentiate w.r.t. q^α:

$$\left\langle \delta \dot{J}_\alpha \right\rangle = \frac{1}{32\pi} \left[\lim_{\Delta u \to \infty} \frac{1}{\Delta u} \int_{\Delta \mathscr{I}} \omega^{a'} \left\{ \boldsymbol{h}^1, \frac{\partial \boldsymbol{h}^1}{\partial q^\alpha} \right\} \mathrm{d}\Sigma_{a'} \right.$$
$$\left. + \lim_{\Delta v \to \infty} \frac{1}{\Delta v} \int_{\Delta \mathscr{H}} \omega^{a'} \left\{ \boldsymbol{h}^1, \frac{\partial \boldsymbol{h}^1}{\partial q^\alpha} \right\} \mathrm{d}\Sigma_{a'} \right]$$

(qualitatively reproduces results of [Isoyama et al., 2018])

Conclusions and future work

- In this talk: flux-balance laws for the action variables $\implies$ evolution for *all* conserved quantities in Kerr
  - Note: calculation doesn't assume these variables!
  - Extends results of [**Grant** & Moxon, 2022] (for scalar fields)
  - "Explains" results of [Isoyama et al., 2018] (Carter constant as coordinate $\implies$ [Mino, 2003]?)



- Future work:
  - Practicality: flux in terms of curvature variables (language of [Isoyama et al., 2018])
  - Second order! (currently sorting out E & L_z)
  - Poisson bracket can be degenerate $\implies$ add spin?